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Embedding Theorem for Lattices with Complementation

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§ 1. Introduction

In this paper, we will show embedding theorem for lattices with complementation. Our aim is twofold; one is of course, a study of lattice theory and the other is a semantical study of some nonclassical logics. We will first explain this relationship in the following.

As is well-known, the following representation theorem for distributive lattices holds, which is proved by Birkhoff and Stone (see [1]); a lattice is distributive if and only if it is isomorphic to a ring of sets. Thus, every distributive lattice can be embedded in a complete distributive lattice, which is made of the power set of a set. This can be proved in the following way. Let $F^*(L)$ be the set of all prime filters of a given distributive lattice L, and $A_{F^*(L)}$ be the set of all subsets of $F^*(L)$. Then, the mapping $h : L \longrightarrow A_{F^*(L)}$ defined by

(1) $h(a) = \{ F \in F^*(L) \ ; \ a \in F \}$ for each $a \in L$,

gives a lattice isomorphism.

This method can be applied to show the embedding theorem for various algebras, which are distributive as lattices. For example, let A be any Heyting algebra. Then, $F^*(A)$ is partially ordered by the set inclusion $\subseteq$ in this case, so we will take the set of all closed subsets of $F^*(A)$, instead of the power set, for $A_{F^*(A)}$. Here, we say that a subset S of a p.o.set M is *closed* if

(2) $x \in S$ and $x \leqq y$ imply $y \in S$.

Then, it can be shown that $A_{F^*(A)}$ is a Heyting algebra and that the mapping defined similarly as (1) is an isomorphism for Heyting algebras.

Here, we will call informally a set F*(A) a *dual space* for a Heyting algebra A. In general, for any given class A of algebras, a class M of structures, called the class of dual spaces for A, will be required to satisfy at least the following ;

1) for each algebra A in A, we can construst a structure X(A) in M,
2) for each X in M, we can construct an algebra A_X in A,
3) there exists an embedding h from A to $A_{X(A)}$, for each A in A.

From a logical point of view, the problem of finding a suitable class of dual spaces for a given class of algebras is closely related to the problem of finding a suitable Kripke-type (or relational) semantics for a given logic. This relationship will be shown schematically as follows:

logic	lattice theory
logic L (e.g. the intuitionistic logic)	the class A of algebras corresponding to L (e.g. the class of Heyting algebras)
Kripke-type semantics for L (e.g. partially ordered sets)	a dual space for A (e.g. partially ordered sets)

completeness theorem for L with respect to the Kripke-type semantics	embedding theorem for each algebra in A, via its dual space

In the following two sections, we will introduce dual spaces for various lattices with complementation and prove the embedding theorem for them, by using Goldblatt's method in [3], which is based on the idea developed in [2]. In §4, we will show how these results can be translated into completeness theorem for corresponding logics.

§ 2. Distributive lattices with complementation

We will consider four kinds of complementations on lattices. Let L be a lattice having the minimum element 0 and the maximum 1, and $'$ be a unary operation on L. We will consider the following conditions on $'$;

(1) $a \cap a' = 0$ for each a,

(2) $a \leqq b'$ implies $b \leqq a'$, for each a, b,

(3) $(a')' \leqq a$ for each a,

(4) $a \cap b = 0$ if and only if $a \leqq b'$, for each a, b.

Then, the operation $'$ is

a *weak pseudo-complementation*, if it satisfies (1) and (2),

a *pseudo-complementation*, if it satisfies (4),

a *quasi-complementation*, if it satisfies (2) and (3),

an *ortho-complementation*, if it satisfies (1), (2) and (3).

We can show easily the following.

LEMMA 1. 1) Any pseudo-complemented lattice is a weakly pseudo-complemented lattice.

2) Any pseudo-complemented lattice satisfying that $(a')' \leqq a$ for each a is an orthomodular lattice, but the converse does not hold.

3) Any ortho-complemented distributive lattice is a Boolean algebra.

In this section we will consider distributive lattices with complementation. We will introduce six kinds of spaces, in the following.

DEFINITION 1. 1) An *S_I-space* is a triple $< X, \leqq ; \perp >$ such that

(1) $< X, \leqq >$ is a nonempty p.o.set with a partial order $\leqq$,

(2) $\perp$ is an irreflexive, symmetric relation on X satisfying that $x \perp y$ and $y \leqq z$ imply $x \perp z$ for every x, y, z.

2) An *S_{II}-space* is a triple $< X, \leqq ; \perp >$ such that

(1) $< X, \leqq >$ is a nonempty p.o.set with a partial order $\leqq$,

(2) $\perp$ is a binary relation on X satisfying the condition that $x \perp y$ if and only if there exist no z's such that $x \leqq z$ and $y \leqq z$.

3) An *S^*-space* is a quadruple $< X, \leqq , * ; \perp >$ such that

(1) $< X, \leqq >$ is a nonempty p.o.set with a partial order $\leqq$,

(2) $*$ is a unary operation on X such that i) $(x^*)^* = x$ and ii) $y \leqq x^*$ implies $x \leqq y^*$,

(3) $\perp$ is a binary relation on X satisfying the condition that $x \perp y$ if and only if $y \not\leqq x^*$.

4) A *T_0-space* is a pair $< X ; \perp >$ of a nonempty p.o.set X and a symmetric relation $\perp$ on X.

5) A *T_I-space* $< X ; \perp >$ is a T_0-space satisfying that $\perp$ is also irreflexive.

6) A T_{III}*-space* is a pair $< X ; \perp >$ of a nonempty set X and a binary relation $\perp$ satisfying the condition that

$$x \perp y \text{ if and only if } x \neq y.$$

In each space S in the above definition, the set X is called *the underlying set of* S. The following can be easily verified.

LEMMA 2. 1) If $< X, \leqq ; \perp >$ is an S_I-space then $< X ; \perp >$ is a T_I-space.

2) Any S_{II}-space is an S_I-space.

3) The relation $\perp$ in any S*-space is symmetric.

4) Any T_{III}-space is a T_I-space.

5) Let $< X, \leqq ; \perp >$ be an S_{II}-space with the trivial order $\leqq$, i.e., $x \leqq y$ implies $x = y$ for each x, y. Then, $< X ; \perp >$ is a T_{III}-space. Conversely, every T_{III}-space supplemented by the trivial order is an S_{II}-space.

6) Let $< X, \leqq , * ; \perp >$ be an S*-space with the trivial order $\leqq$ such that $x^* = x$ holds for each x. Then $< X ; \perp >$ is a T_{III}-space. Conversely, each T_{III}-space $< X ; \perp >$ supplemented by the trivial order and a unary relation * satisfying $x^* = x$, is an S*-space.

Let $\perp$ be any fixed binary relation on a set X. For each subset S of X, define a subset $S^\perp$ of X by

$$S^\perp = \{ x \in X ; \text{ for each } y, \ y \in S \text{ implies } x \perp y \}.$$

For any p.o.set X, a subset S of X is said to be *closed* if

$$x \in S \text{ and } x \leqq y \text{ imply } y \in S.$$

LEMMA 3. Let S be an S_I-space or an S_{II}-space or an S*-space, with the underlying set X. If S, S_1 and S_2 are closed subsets of X then S , $S_1 \cup S_2$ and $S_1 \cap S_2$ are also closed. Moreover, if S_i is a closed subset of X for each $i \in I$ then both $\bigcup_{i \in I} S_i$ and $\bigcap_{i \in I} S_i$ are also closed.

LEMMA 4. For each closed subsets S_1 and S_2 of X,

$$S_1 \cup S_2 = (S_1^{\perp} \cap S_2^{\perp})^{\perp}$$

holds, if S is either an S*-space or a T_{III}-space.

Proof. Since $S_1 \cup S_2 \subseteq (S_1^{\perp} \cap S_2^{\perp})^{\perp}$ holds always, we have only to show the converse inclusion. Suppose that S (= $< X, \leqq , * ; \perp >$) is any S*-space. We assume that x is in $(S_1^{\perp} \cap S_2^{\perp})^{\perp}$. Then, for every $z \in S_1^{\perp} \cap S_2^{\perp}$, $x \perp z$, i.e., $z \not\leqq x^*$. Hence, $x^* \notin S_1^{\perp} \cap S_2^{\perp}$. So, either $x^* \notin S_1^{\perp}$ or $x^* \notin S_2^{\perp}$. If $x^* \notin S_1^{\perp}$ then $x^* \perp u$ does not hold for some $u \in S_1$. This means that $u \leqq (x^*)^* = x$ for some $u \in S_1$. Hence, $x \in S_1$. Similarly, if $x^* \notin S_2^{\perp}$ then $x \in S_2$. Therefore, $x \in S_1 \cup S_2$. By using Lemma 2 6), we can show our lemma when S is a T_{III}-space. We remark here that every subset of X is closed if $< X, \leqq >$ is a p.o.set with the trivial order $\leqq$.

LEMMA 5. 1) Let S be an S_I-space (or an S_{II}-space or an S*-space), with the underlying set X. Then, the set A_S of all closed subsets of X forms a complete, weakly pseudo-complemented (or, pseudo-complemented or quasi-complemented, respectively) distributive lattice with respect to $\cup$, $\cap$ and $\perp$.
2) Let S be a T_I-space (or a T_{III}-space) with the underlying set X. Then, the power set B_S of X forms a complete, weakly pseudo-complemented (or, ortho-complemented, respectively) distributive lattice with respect to $\cup$, $\cap$ and $\perp$.

Let L be any lattice with the complementation $'$. A nonempty subset F of L is a *filter* (of L), if

(1) $a \in F$ and $a \leq b$ imply $b \in F$,

(2) $a \in F$ and $b \in F$ imply $a \cap b \in F$.

A filter F of L is *proper* if F is a proper subset of L. Moreover, a proper filter F is *prime* if

(3) $a \cup b \in F$ implies either $a \in F$ or $b \in F$.

The set of all proper filters of L is denoted by F(L) and the set of all prime filters of L is denoted by F*(L). Next, we will define a binary relation $\perp$ on F(L) by

$F \perp G$ if and only if $a \in F$ and $a' \in G$ for some $a \in L$.

LEMMA 6. 1) If L is a weakly pseudo-complemented distributive lattice then $\langle F^*(L), \subseteq ; \perp \rangle$ is an S_I-space, and hence $\langle F^*(L) ; \perp \rangle$ is a T_I-space. Similarly, if L is a pseudo-complemented distributive lattice then $\langle F^*(L), \subseteq ; \perp \rangle$ is an S_{II}-space.

2) If L is a quasi-complemented distributive lattice then $\langle F^*(L), \subseteq, * ; \perp \rangle$ is an S*-space, where * is defined by

$$F^* = \{ x ; \ x' \notin F \}.$$

3) If L is an ortho-complemented distributive lattice then $\langle F^*(L) ; \perp \rangle$ is a T_{III}-space.

Proof. 1) We will show only that $\perp$ is an irreflexive, symmetric relation on F*(L). Suppose that $F \in F^*(L)$ and $a, a' \in F$ for some a. Then $a \cap a' = 0 \in F$. But this contradicts the fact that F is proper. Hence $F \perp F$ does not hold. Next suppose that $F \perp G$. Then there exists $a \in L$ such that $a \in F$ and $a' \in G$. Since $a \leq (a')'$ holds in every weakly pseudo-complemented lattice, $a' \in G$ and $(a')' \in F$ hold. Hence, $G \perp F$. Next suppose that L is a pseudo-complemented lattice. We will show that $F \perp G$ if and only if there are no H's in $F^*(L)$ such that $F \subsetneq H$ and $G \subsetneq H$. We assume first that $F \perp G$, $F \subsetneq H$ and $G \subsetneq H$ for some $H \in F^*(L)$. Then for some $a \in L$, $a \in F \subsetneq H$ and $a' \in G \subsetneq H$. Therefore, $a \cap a' = 0 \in H$. But, this is a contradiction. Conversely, suppose that $F \perp G$ does not hold. Let E be the filter generated by $F \cup G$. If E is not proper then there exist $a \in F$ and $b \in G$ such that $a \cap b = 0$. Since L is a pseudo-complemented lattice, $b \leqq a'$ follows from this. Thus, $a \in F$ and $a' \in G$. This implies $F \perp G$. But this is a contradiction. So, E is proper. Therefore, E can be extended to a prime filter H, which contains both F and G.

2) We can show easily that $(F^*)^* = F$ and that $G \subsetneq F^*$ implies $F \subsetneq G^*$. Next we will show that $F \perp G$ if and only if $G \not\subseteq F^*$. Suppose that $F \perp G$. Then $G \perp F$, so $a \in G$ and $a' \in F$ for some a. Hence, $a \in G - F^*$. The converse of these implications holds also.

3) We will show that $F \perp G$ if and only if $F \neq G$. By 1), $\perp$ is irreflexive, so $F \perp G$ implies $F \neq G$. Suppose that $F \neq G$. Let $a \in F - G$. Since L is a Boolean algebra by Lemma 1 3), $a \cup a' = 1$ holds. On the other hand, since G is prime and $a \cup a' = 1 \in G$, $a' \in G$. Thus, $F \perp G$.

For each complemented distributive lattice L, the dual space defined in Lemma 6 is denoted by S(L). Combining Lemma 5 with Lemma 6, we have the following.

THEOREM 7 (Embedding theorem for complemented distributive lattices) Let L be a weakly pseudo-complemented distributive lattice. Then L can be embedded in a complete, weakly pseudo-complemented distributive lattice $A_{S(L)}$. In fact, the mapping $h : L \longrightarrow A_{S(L)}$ defined by

$$h(a) = \{ F \in F^*(L) \ ; \ a \in F \} \qquad \text{for each } a \in L,$$

gives a lattice isomorphism. (The mapping h can be considered also as a lattice isomorphism from L to $B_{S(L)}$, since $A_{S(L)}$ is a subalgebra of $B_{S(L)}$ in this case.) Similar result holds also for pseudo-complemented or quasi-complemented distributive lattices. Similarly, the mapping $h : L \longrightarrow B_{S(L)}$ defined as the above gives a lattice isomorphism, when L is an ortho-complemented lattice.

Proof. It suffices to show that h is an isomorphism. Here we will show only that $h(a') = h(a)^{\perp}$. Let $F \in h(a')$. Then, $a' \in F$. Thus, $F \perp G$ for every $G \in h(a)$. Conversely, suppose that $F \notin h(a')$. We will show that there exists a prime filter H in h(a) such that $F \perp H$ does not hold. Notice here that $a > 0$, since otherwise $a' = 1 \in F$. Now define the set I by

$$I = \{ G \ ; \ G \text{ is a proper filter such that } a \in G \text{ and } F \perp G \text{ does not hold} \}.$$

Let $F_a = \{ x \ ; \ a \leqq x \}$. Then, $F_a \in I$. For, if $F \perp F_a$ then $b \in F$ and $b' \in F_a$ for some b. So, $a \leqq b'$ and therefore $b \leqq a'$. Hence, $a' \in F$. But this is a contradiction. Thus, I is nonempty and inductive. So, there exists a maximal element H in I by Zorn's Lemma. Moreover, we can show that H is prime.

§ 3. Nondistributive lattices with complementation

In this section, we will deal with dual spaces for non-distributive lattices with complementation. We remark here that prime filters of lattices does not work well in nondistributive lattices. So, it is necessary to modify our approach developed in the previous section.

DEFINITION 2. 1) A *U_0-space* is a triple $\langle X, \leq ; \perp \rangle$ such that

(1) $\langle X, \leq \rangle$ is a nonempty meet-semilattice with respect to the partial order $\leq$,

(2) $\perp$ is a symmetric relation satisfying that for each x, y, $z \in X$,

i. if $x \perp y$ and $y \leq z$ then $x \perp z$,

ii. if $x \perp y$ and $x \perp z$ then $x \perp (y \cap z)$.

2) A *U_I-space* is a U_0-space $\langle X, \leq ; \perp \rangle$ such that $\perp$ is also irreflexive.

3) A *U_{II}-space* is a triple $\langle X, \leq ; \perp \rangle$ such that

(1) $\langle X, \leq \rangle$ is a nonempty meet-semilattice with respect to the partial order $\leq$,

(2) $\perp$ is a binary relation on X satisfying the condition that

i. $x \perp y$ if and only if there exist no z's such that $x \leq z$ and $y \leq z$,

ii. if $x \perp y$ and $x \perp z$ then $x \perp (y \cap z)$.

Similarly as Lemma 2, we have the following.

LEMMA 8. 1) If $\langle X, \leq ; \perp \rangle$ is a U_0-space (or a U_I-space) then $\langle X ; \perp \rangle$ is a T_0-space (or a T_I-space).

2) Any U_{II}-space is a U_I-space.

Let $< X, \leqq ; \perp >$ be a U_0-space. Then, a subset S of X is said to be $\cap$-*closed* if it is closed and if $x, y \in S$ implies $x \cap y \in S$. Also, a subset S of X is said to be *regular* if $(S^{\perp})^{\perp} = S$. For each U_0-space S with the underlying set X, the set of all $\cap$-closed subsets (or the set of all regular subsets, or the set of all $\cap$-closed , regular subsets) of X is denoted by C_S (or D_S or E_S, respectively). Next, we will define $S_1 \vee_1 S_2$ and $S_1 \vee_2 S_2$ for each subset S_1 and S_2 of X by

$$S_1 \vee_1 S_2 = \begin{cases} S_2 & \text{if } S_1 = \emptyset, \\ S_1 & \text{if } S_2 = \emptyset, \\ \{ x \; ; & y \cap z \leqq x \text{ for some } y \in S_1 \text{ and } z \in S_2 \} \\ & \text{otherwise}, \end{cases}$$

$$S_1 \vee_2 S_2 = (S_1^{\perp} \cap S_2^{\perp})^{\perp}.$$

Moreover, for each set $\{S_i\}_{i\in I}$ of subsets of X, define

$$\bigvee_{i\in I}{}_1 S_i = \{ x \; ; \text{ for some } m \text{ and some } i_1,\dots,i_m \in I, \; y_{i_j} \in S_{i_j} \; (j = 1,\dots,m) \text{ and } y_{i_1} \cap \dots \cap y_{i_m} \leqq x \},$$

$$\bigvee_{i\in I}{}_2 S_i = (\bigcap_{i\in I} S_i^{\perp})^{\perp}.$$

LEMMA 9. Let $< X, \leqq ; \perp >$ be any U_0-space.

1) If S, S_1 and S_2 are $\cap$-closed subsets of X then S , $S_1 \cap S_2$ and $S_1 \vee_1 S_2$ are also $\cap$-closed. Moreover, if S_i is a $\cap$-closed subset of X for each $i \in I$ then both $\bigcap_{i\in I} S_i$ and $\bigvee_{i\in I}{}_1 S_i$ are also $\cap$-closed.

2) If S, S_1 and S_2 are regular then S , $S_1 \cap S_2$ and $S_1 \vee_2 S_2$ are also regular. Moreover, if S_i is regular for each $i \in I$ then both $\bigcap_{i\in I} S_i$ and $\bigvee_{i\in I}{}_2 S_i$ are also regular.

We can see that $S_1 \vee_1 S_2$ (or $S_1 \vee_2 S_2$) is the smallest $\cap$-closed (or regular) subset containing both S_1 and S_2, if both S_1 and S_2 are $\cap$-closed (or regular, respectively).

LEMMA 10. 1) Let S be a U_I-space (or a U_{II}-space). Then, C_S forms a complete, weakly pseudo-complemented (or, pseudo-complemented) lattice with respect to $\vee_1$, $\cap$ and $^{\perp}$.

2) Let S be a T_0-space (or a T_I-space). Then, D_S forms a complete, quasi-complemented (or ortho-complemented) lattice with respect to $\vee_2$, $\cap$ and $^{\perp}$.

3) Let S be a U_0-space (or a U_I-space, or a U_{II}-space). Then, E_S forms a complete, quasi-complemented lattice (or ortho-complemented lattice, or pseudo-complemented lattice satisfying $(a')' \leqq a$, respectively) with respect to $\vee_2$, $\cap$ and $^{\perp}$.

We can observe that the set F(L) of all proper filters of a lattice L forms a meet-semilattice with respect to the set inclusion. Now, we have the following lemma, which corresponds to Lemma 6.

LEMMA 11. 1) If L is a weakly pseudo-complemented (or, a pseudo-complemented, or a quasi-complemented) lattice, then $\langle F(L), \subseteq ; \perp \rangle$ is a U_I-space (or a U_{II}-space, or a U_0-space, respectively).

2) If L is a quasi-complemented (or an ortho-complemented) lattice then $\langle F(L) ; \perp \rangle$ is a T_0-space (or a T_I-space).

Proof. We will show only that $F \perp G$ and $F \perp H$ imply $F \perp (G \cap H)$ for every F, G, H $\in$ F(L). By the assumption, there exist a and b such that $a \in F$, $a' \in G$, $b \in F$ and $b' \in H$. Then, clearly $a \cap b \in F$ and $a' \cup b' \in G \cap H$. Notice here that $a' \cup b' \leqq (a \cap b)'$ holds, whenever the condition

$$a \leqq b' \text{ implies } b \leqq a' \quad \text{for each } a, b$$

is satisfied. Hence, $a \cap b \in F$ and $(a \cap b)' \in G \cap H$. Thus, $F \perp (G \cap H)$. By using Lemma 8 1), 2) can be derived from 1).

THEOREM 12 (Embedding theorem for complemented, nondistributive lattices) 1) Let L be a weakly pseudo-complemented (or a pseudo-complemented) lattice. Then, L can be embedded in a complete, weakly pseudo-complemented (or pseudo-complemented) lattice $C_{S(L)}$. In fact, the mapping $h' : L \longrightarrow C_{S(L)}$ defined by

$$h'(a) = \{ F \in F(L) \; ; \; a \in F \} \quad \text{for each } a \in L,$$

gives a lattice isomorphism.

2) Similarly, the mapping $h' : L \longrightarrow E_{S(L)}$ gives a lattice isomorphism, when L is a quasi-complemented lattice or an ortho-complemented lattice, or a pseudo-complemented lattice satisfying $(a')' \leqq a$.

3) When L is either a quasi-complemented or an ortho-complemented lattice, $E_{S(L)}$ is a subalgebra of $D_{S(L)}$. Thus, h' can be also regarded as a lattice isomorphism of L into $D_{S(L)}$, in these cases.

The above embedding theorem for each ortho-complemented lattice L into another lattice $D_{S(L)}$ is an algebraic version of the result proved by Goldblatt [3].

§ 4. Completeness theorem

By using these embedding results, we can introduce Kripke-type semantics for logics corresponding to these lattices and show the completeness theorem for them, as we have mentioned in § 1. Here, we will take the *weakly pseudo-complemented logic* as an example. Other logics can be treated quite similarly.

We will take $\wedge$, $\vee$ and $\neg$ as logical connectives. Formulas can be defined in the usual way. Let L be any weakly pseudo-complemented lattice. An *assignment* f of L is a mapping from the set of propositional variables to L. Then f can be extended to a mapping from the set of formulas to L, following the requirements;

(1) $f(A \wedge B) = f(A) \cap f(B)$,

(2) $f(A \vee B) = f(A) \cup f(B)$,

(3) $f(\neg A) = f(A)'$.

We say that a formula B is *derivable from* formulas $A_1, \ldots , A_m$ in the *weakly pseudo-complemented logic* L_{WP} and write

$$A_1, \ldots , A_m \vdash B,$$

if for every weakly pseudo-complemented lattice L and every assignment f of L,

$$f(A_1) \cap \ldots \cap f(A_m) \leqq f(B) \quad (\text{ or, } \ f(B) = 1, \text{ when } m = 0),$$

holds. (of course, it is possible to define the logic L_{WP} in a purely syntactical way. But this is not essential in the following argument.)

Next, we will introduce a Kripke-type semantics for L_{WP}. We call any U_I-space an *L_{WP}-structure*. A *valuation* $\models$ on an L_{WP}-structure $< X, \leqq ; \perp >$ is a relation between the set X and the set of propositional variables satisfying that for each a, b $\in$ X and each propositional variable p,

(1) if $a \models p$ and $a \leqq b$ then $b \models p$,

(2) if $a \models p$ and $b \models p$ then $a \cap b \models p$.

Each valuation $\models$ can be extended to a relation between X and the set of formulas by the requirements;

(3) $a \models \neg A$ if and only if $b \not\models A$ for each b such that $a \not\leq b$,

(4) $a \models A \wedge B$ if and only if $a \models A$ and $a \models B$,

(5) $a \models A \vee B$ if and only if i) for some b, c such that $b \cap c \leqq a$, $b \models A$ and $c \models B$, or ii) $a \models A$, or iii) $a \models B$

We can show that for each formula A,

(6) if $a \models A$ and $a \leqq b$ then $b \models A$,

(7) if $a \models A$ and $b \models A$ then $a \cap b \models A$.

This can be proved quite similarly as Lemma 9 1). We say that a formula A is a *semantical consequence* of formulas $A_1, \ldots, A_m$ with respect to L_{WP}-structures, if for each L_{WP}-structure $< X, \leqq ; \perp >$ and each valuation $\models$ on it, $a \models A_1$, ... and $a \models A_m$ imply $a \models B$ for every $a \in X$. By using Lemma 10 1) and Lemma 11 1), we have the following theorem.

THEOREM 13 (Completeness theorem for the weakly pseudo-complemented logic L_{WP}) For each formula $A_1, \ldots, A_m$, B, B is derivable from $A_1, \ldots, A_m$ in L_{WP} if and only if B is a semantical consequence of $A_1, \ldots, A_m$ with respect to L_{WP}-structures.

A different approach to Kripke-type semantics for (non-distributive) logics with some weak negations will be seen in [5]. By combining the method in [5] with the method developed in this paper, we have obtained the completeness theorem for the classical logic without the contraction rules in [6]. An interesting problem which remains open is to find a suitable class of dual spaces for orthomodular lattices, or equivalently, to find a suitable Kripke-type semantics for the *orthomodular logic*.

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